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Consistency and correctness evaluation of shear deformable anisoparametric formulations

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Abstract

The first two of the quartet, *completeness*, *continuity*, *consistency* and *correctness* conditions, required for the development of robust displacement based finite elements are well understood as far as the anisoparametric elements are concerned. When such elements are formulated (primarily for accuracy due to their improved consistent load and mass capabilities) by incorporating transverse shear deformation effect, it becomes necessary to study the next two of the quartet—(field) consistency and (variational) correctness aspects and to explore further their susceptibility for spurious effects such as locking, stress oscillations etc. Such a study on a typical anisoparametric element formulation is presented in this paper. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Most general purpose finite element computing is now based on shear deformable beam (Timoshenko) and plate/shell (Mindlin) elements. These are usually based on simple isoparametric shape functions. Problems like locking that plagued such formulations have now been eliminated using reduced integration or other equivalent field-consistent approaches (Prathap, 1993).

The simplest beam element in this class is a two-noded element using linear C^0 functions for the transverse displacement w and face rotation θ. It is capable of representing a state of constant bending moment and constant shear force within each element. The convergence characteristics of this element depends on this fact and also on the way consistent distribution of applied forces and mass matrices are performed. Thus, a large number of elements must be used (Kant and Marur, 1991) if high accuracy is

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needed in an application. This shortcoming is particularly evident in applications such as dynamics or buckling, or even statics under rapidly varying distributed loading, where consistent mass, geometric stiffness matrices and load vector are not of high accuracy due to the simple linear representation for w.

This state of affairs can be improved if one resorts to what is called an ‘anisoparametric formulation’ (Tessler and Dong, 1981). Here, one can adopt a higher order representation for w, which is the factor that determines the accuracy of the consistently derived load and mass matrices while retaining a linear representation for θ.

As the completeness and continuity of such class of shear deformable elements are well-known (Davis et al., 1972; Nickel and Secor, 1972; Thomas et al., 1973), it is also perceived to be important to explore the (field) consistency and (variational) correctness aspects of such formulations to understand the behaviour related to locking, stress oscillations, etc. Such an analysis of a typical shear deformable anisoparametric element is presented here.

2. Element definition

A two-noded element with the following six degrees of freedom is considered:

$$d = [w_1\theta_1\beta_1 w_2\theta_2\beta_2]^t, \tag{1}$$

where w is the transverse displacement, θ is the rotation of cross-section and β is the rotation of the neutral axis defined by the first derivative of w represented as $w_{,x}$.

The transverse displacement in an element of length L $(=2l)$ is expressed using Hermitian polynomial as

$$w = a_0 + a_1\left(\frac{x}{l}\right) + a_2\left(\frac{x}{l}\right)^2 + a_3\left(\frac{x}{l}\right)^3. \tag{2}$$

The four constants of the above equation can be expressed in terms of the nodal degrees of freedom as

$$a_0 = \frac{(w_1 + w_2)}{2} - \frac{(\beta_2 - \beta_1)l}{4}, \tag{3a}$$

$$a_1 = \frac{3(w_2 - w_1)}{4} - \frac{(\beta_1 + \beta_2)l}{4}, \tag{3b}$$

$$a_2 = \frac{(\beta_2 - \beta_1)l}{4} \tag{3c}$$

$$a_3 = \frac{(w_1 - w_2)}{4} + \frac{(\beta_1 + \beta_2)l}{4}. \tag{3d}$$

The rotation of cross section is expressed by a linear polynomial as

$$\theta = b_0 + b_1\left(\frac{x}{l}\right), \tag{4}$$

where $b_0 = (\theta_1 + \theta_2)/2$ and $b_1 = (\theta_2 - \theta_1)/2$, and θ_1 and θ_2 are defined at $x = -l$ and $x = l$ respectively.

3. Field-consistency interpretation

An examination of this element from field-consistency considerations (Prathap, 1993) is carried out now. The consistency paradigm argues that spurious effects like locking, delayed convergence and stress-oscillations can occur in a problem only if the shear strain field which is used to compute the shear strain energy generates spurious constraints. The bending and shear strains, for the interpolation scheme described by Eqs. (2) and (4), can be described as

$$\chi = \frac{b_1}{l}, \tag{5a}$$

$$\gamma = \left(b_0 - \frac{a_1}{l} - \frac{a_3}{l}\right) + \left(b_1 - \frac{2a_2}{l}\right)\left(\frac{x}{l}\right) + \left(\frac{a_3}{l}\right)\left[1 - 3\left(\frac{x}{l}\right)^2\right] \tag{5b}$$

and the resulting shear strain energy can be expressed as

$$U_S = \frac{1}{2}kGA(2l)\left[\left(b_0 - \frac{a_1}{l} - \frac{a_3}{l}\right)^2 + \frac{1}{3}\left(b_1 - \frac{2a_2}{l}\right)^2 + \frac{4}{5}\left(\frac{a_3}{l}\right)^2 \Big/ 5\right]. \tag{5c}$$

The constraints that would emerge at the thin-beam or Kirchhoff limit are:

$$b_0 - \frac{a_1}{l} - \frac{a_3}{l} \rightarrow 0, \tag{6a}$$

$$b_1 - \frac{2a_2}{l} \rightarrow 0 \tag{6b}$$

$$a_3 \rightarrow 0. \tag{6c}$$

In the field-consistency terminology, Eqs. (6a) and (6b) reflect physically meaningful constraints, showing a consistent balance of terms from the w and θ fields. The only cause for concern is the constraint appearing in Eq. (6c), as it only comprises terms from the w field—an inconsistent constraint (Prathap, 1993).

The effect of a_3 on the mechanics of the element would be studied *a posteriori* through a numerical experiment. The element stiffness matrix, consistent mass matrix and load factors are generated using usual procedures and are not elaborated here. Exact integration is used for shear terms.

Let this element version be designated as ORIG, to denote that it is the original element without any extra-variational manipulations like reduced integration or strain smoothing.

4. Consistent-(variationally) correct element

It would be useful to examine a strain-smoothed element which will be field-consistent in the strictest sense in which it is defined by Prathap (1993). It is required to compute an 'assumed' strain for this, which will be of the form:

$$\gamma_{CC} = c_0 + c_1\left(\frac{x}{l}\right). \tag{7}$$

A variationally correct reconstitution of this strain-field can be made using the orthogonality condition that emerges from the Hu–Washizu theorem (Prathap, 1993):

$$\int \delta\gamma_{CC}^t(\gamma_{CC} - \gamma)\mathrm{d}x = 0. \tag{8}$$

This gives the consistent and correct strain field as

$$\gamma_{CC} = \left(b_0 - \frac{a_1}{l} - \frac{a_3}{l}\right) + \left(b_1 - \frac{2a_2}{l}\right)\left(\frac{x}{l}\right). \tag{9}$$

Let the element whose shear related stiffnes matrix is based on this field-consistent strain, γ_{CC}, be refered to as the CC element to indicate that the Consistent strain-field has been reconstituted in a variationally Correct way.

5. Consistent-(variationally) incorrect element

A commonly used procedure for deriving a smoothed strain is to apply the least squares smoothing operation to the displacement function and then to derive the smoothed strain from this (Jayachandrabose and Kirkhop, 1984). Thus, in this instance, a smoothed w^* is obtained from w using

$$\delta\int(w^* - w)^2\,\mathrm{d}x = 0. \tag{10}$$

If a shear strain is then computed using $w_{,x}^*$ it can then be expressed as

$$\gamma_{CI} = \left(b_0 - \frac{a_1}{l} - \frac{a_3}{l} + \underline{\frac{2a_3}{5l}}\right) + \left(b_1 - \frac{2a_2}{l}\right)\left(\frac{x}{l}\right). \tag{11}$$

It must be noted that γ_{CI} differs from γ_{CC} by the underlined term in Eq. (11)–$(2a_3/5l)$. Moreover, the reconstitution procedure used here has violated the variational correctness norm, which can be strictly met only if the orthogonality condition described in Eq. (8) is satisfied. The difference seen between γ_{CC} and γ_{CI} is due to the violation of this correctness requirement. Its implication is worked out in the next section.

Let the element whose shear related stiffness matrix is based on the smoothed shear strain, derived in Eq. (11) be called the CI element to indicate that the Consistent strain field has been reconstituted in a variationally Incorrect way.

Table 1
Data for test problem

$L = 10$ in; $b = 1$ in; $d = 0.1$ in
$E = 3 \times 10^7$ lb/in^2
$\nu = 0.3$
$G = 1.1538 \times 10^7$ lb/in^2
$k = 1.2$
$\rho = 0.733 \times 10^{-3}$ lb sec^2/in^4
Triangularly varying load:
$q = 60$ lb at $X = 0$; $q = 0$ at $X = L$
Note:
$L = 2l$
$X = (1 + x/l)L/2$; $0 \le X \le L$; $-l \le x \le l$

6. Analytical predictions

The test problem given in Table 1 is carefully chosen to highlight the performance capabilities of various versions of this element (ORIG, CC and CI) and which allows us to relate their performance to predictions derived from the basic paradigms involved in understanding their behaviour.

The intensity of loading can be described by the function

$$q(X) = \frac{q(L - X)}{L}. \tag{12}$$

Simple engineering analysis allows the computation of shear force and bending moment as

$$Q(X) = \frac{q(L - X)^2}{2L} \tag{13}$$

$$M(X) = \frac{q(L - X)^3}{6L}. \tag{14}$$

The shear force in the beam, for a given load of 60 lb, will vary quadratically from 0 at the free-end to 300 at the fixed-end and the bending moment will vary cubically from 0 to 1000 over the same range.

The bending strain/moment can be computed only as a constant in an element using ORIG formulation. Recent evidence (Prathap, 1994) indicates that these values will be a least squares accurate fit of the actual loading. The bending moment for this problem will therefore be computed in a step-wise fashion. As the beam is thin, with an aspect ratio of 100, the main action is bending; therefore, convergence of the tip deflection will be determined mainly by the rate at which the computed bending moment pattern approaches the actual cubic variation.

It is important to anticipate analytically the response of the three element versions to the given loading configuration. As the computed shear strain/force can vary quadratically within each ORIG element, as shown by Eq. (5b), this element should be able to recover the shear force variation exactly, even with a single element.

On the other hand, the CC and CI elements will be able to capture this variation in a linear sense only (Eqs. (9) and (11)). However, the CI element is expected to be in error in some manner because of the variational incorrectness introduced. This will be examined with a single element model of CC and CI versions through the test problem.

Table 2
Convergence of normalised tip deflections of a cantilever ($w_n = 30wEI/qL^4$)

No. of elements	ORIG	TB2
1	0.625	1.250
2	0.899	1.094
3	0.954	1.044
4	0.974	1.026

Let Q_{an} be the analytical solution of shear force variation and let Q_{orig}, Q_{CC} and Q_{CI} represent the shear force values of the corresponding element versions. It can be shown that

$$Q_{\mathrm{an}} = \frac{ql}{4}\left[\frac{4}{3} - 2\left(\frac{x}{l}\right) - \frac{1}{3}\left\{1 - 3\left(\frac{x}{l}\right)^2\right\}\right], \tag{15}$$

$$Q_{\mathrm{orig}} = Q_{\mathrm{CC}} + kGA\left(\frac{a_3}{l}\right)\left[1 - 3\left(\frac{x}{l}\right)^2\right], \tag{16}$$

$$Q_{\mathrm{CC}} = kGA\left[\left(b_0 - \frac{a_1}{l} - \frac{a_3}{l}\right) + \left(b_1 - \frac{2a_2}{l}\right)\left(\frac{x}{l}\right)\right] = \frac{ql}{4}\left[\frac{4}{3} - 2\left(\frac{x}{l}\right)\right] \tag{17}$$

$$Q_{\mathrm{CI}} = Q_{\mathrm{CC}} + kGA\left(\frac{2a_3}{5l}\right), \tag{18}$$

where A is the area of cross-section and k is the shear correction factor. It can be argued, from Eqs. (15) and (16), that the finite element procedure will choose $kGA(a_3/l)$ as being equal to $(-ql/12)$. Substituting this value in Eq. (18), it is clear that for the present test problem, one gets

$$Q_{\mathrm{CI}} = Q_{\mathrm{CC}} - 10. \tag{19}$$

Thus, the shear force from the variationally incorrect CI element will differ from that predicted by the

Table 3
Convergence of flexural frequencies (rad/sec) of a simply supported beam

Mode #	Element type	No. of elements					Exact[a]
		2	3	4	8	16	
1	ORIG	639.6	603.4	591.4	580.0	577.2	
	TB2	809.0	664.8	623.6	587.6	579.0	576.385
2	ORIG	—	—	2257	2364	2319	
	TB2	—	—	3233	2493	2349	2305.540

[a] Based on Euler–Bernoulli's classical beam theory.

Table 4
Convergence of fundamental thickness shear frequency (rad/sec) of a simply supported beam

Element type	No. of elements				Exact
	1	2	3	4	
ORIG	0.3967×10^7	0.3967×10^7	0.3967×10^7	0.3967×10^7	0.3967×10^7
TB2	0.3967×10^7	0.3967×10^7	0.3967×10^7	0.3967×10^7	

CC element by a constant shift of 10, if one element idealisation is used. Numerical experiments would

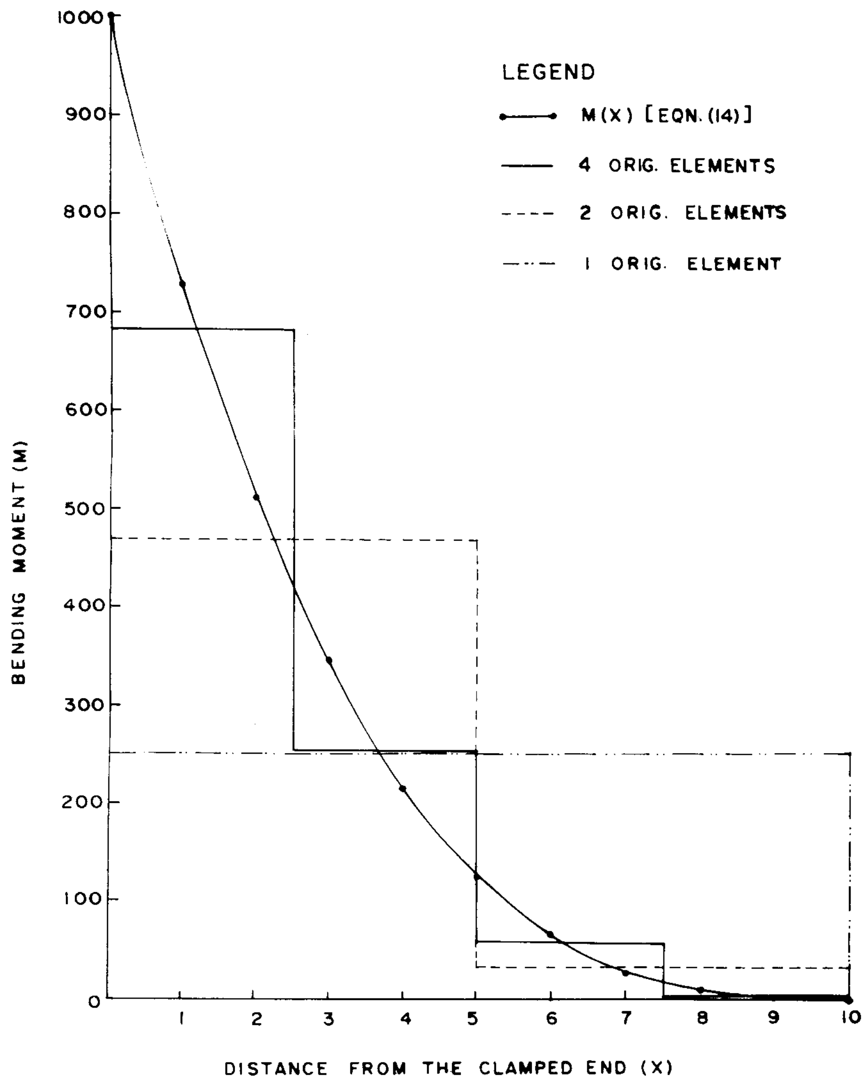

Fig. 1. Bending moment distribution of a cantilever.

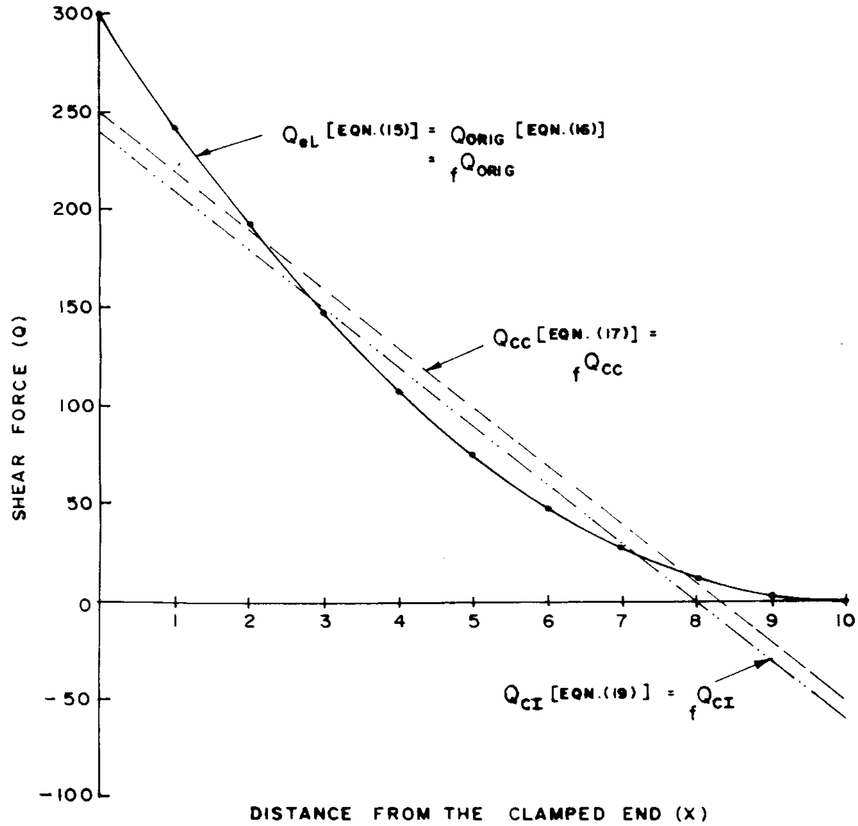

Fig. 2. Shear force distribution for one element idealization of a cantilever.

confirm this later. The CC element model will pick up the correct least squares linear fit of the actual variation.

7. Numerical experiments

The test problem is analysed using these three element models. Results from a conventional isoparametric two-noded beam element based on Timoshenko theory (TB2) are also given in Tables 2–4 for comparison.

Convergence of tip deflections of a cantilever towards the analytical solution, as the number of elements is increased from 1 to 4, is shown in Table 2. Similarly, the proper convergence of flexural frequencies of the ORIG element, for a simply supported beam, can be seen in Table 3. Also, a single ORIG element is seen to pick up the fundamental thickness shear frequency exactly, for the same beam in Table 4.

The actual variation and the stair-step variation of bending moment captured by 1, 2 and 4 elements are shown in Fig. 1. The exact variation is therefore approached closely as more elements are used. This reinforces the earlier observation about the correlation between the convergence of tip deflection to the analytical solution and that of bending moment to the actual cubic variation—both converging with four ORIG elements.

The shear force variation with one element idealisation is shown in Fig. 2. The subscript ‘f’ used in this figure distinguishes results obtained through actual finite element computation of the three element versions from those predicted *a priori* through the error models (Eqs. (16) and (18)). This figure shows that these predictions are confirmed exactly.

8. Classification of constraints in anisoparametric formulations

In the light of the results of the ORIG element, the inconsistent constraint of Eq. (6c) needs to be reinterpreted *a posteriori*. As the ORIG element could compute accurate displacements, frequencies, bending moment and shear force distribution, without any of the ill-effects, it turns out that $a_3 \rightarrow 0$ cannot be spurious, in spite of being inconsistent. This is a case different from all other inconsistent constraints studied so far (Prathap, 1994) which spuriously constrained the unconstrained strain-field.

The reason behind the behaviour of the constraint $a_3 \rightarrow 0$ could be explained by studying the unconstrained strain-field Eq. (5a). In this formulation, the bending strain which is described solely by θ, does not contain the other field-variable, w, to be spuriously constrained by a_3. This results in robust predictions by the ORIG element, formulated using exact integration.

This observation leads to the classification of constraints into two groups:

> *Whenever a degree of freedom, which introduces inconsistent constraint(s), also participates in other unconstrained/constrained strain-fields, then the inconsistent constraint(s) would turn spurious, resulting in effects such as locking etc. Such constraints are termed as ‘Inconsistent–Spurious Constraints’ (ISC).*
> *Otherwise, the inconsistent constraint remains nonspurious leading to a robust element formulation even with full-integration of the constrained energy terms. Such constraints can be designated as ‘Inconsistent–Nonspurious Constraints’ (INC).*

9. Constraints of anisoparametric elements—revisited

Now, it would be quite instructive to explore the nature of constraints emerging from various anisoparametric formulations published so far.

9.1. Inconsistent nonspurious constraints (INC)

Heyliger and Reddy (1988) proposed a higher order anisoparametric formulation, with cubic w and linear θ. Full integration for shear had been employed without any shear locking effects.

Now, let the constraints (Eqs. (6a), (6b) and (6c)) be substituted with the values of constants given by Eqs. (3a), (3b), (3c), (3d) and (4) and one gets the following equations:

$$\frac{(\theta_1 + \theta_2)}{2} = \frac{(w_2 - w_1)}{L}, \tag{20}$$

$$\theta_1 = \beta_1,$$

$$\theta_2 = \beta_2, \tag{21}$$

$$\frac{(\beta_1+\beta_2)}{2}=\frac{(w_2-w_1)}{L}, \tag{22}$$

where $L = 2l$.

It is interesting to note that the three constraints that have been extracted from the shear strain by Heyliger and Reddy (1988) are exactly the same as those given by Eqs. (21) and (22). Moreover, it can be seen that when Eq. (21) is substituted into Eq. (20), the constraint given by Eq. (22) is obtained.

It is the nature of Eq. (22)—an INC (*once the higher order and the nonlinear strain terms are dropped*) which would enable the full integration of shear related terms to yield results without any locking effects.

9.2. Inconsistent spurious constraints (ISC)

Now, this class of constraints which produced locking and stress oscillations in C^0 elements (Prathap, 1993) could be studied for their behaviour in anisoparametric elements.

Carnegie et al. (1969) proposed a four noded Timoshenko element in which both the transverse displacement and face rotation are expanded using cubic polynomial as

$$w=a_0+a_1x+a_2x^2+a_3x^3 \tag{23}$$

$$\theta=b_0+b_1x+b_2x^2+b_3x^3, \tag{24}$$

resulting in a shear field as,

$$\gamma=(a_1+b_0)+(2a_2+b_1)x+(3a_3+b_2)x^2+b_3x^3, \tag{25}$$

where $b_3 \to 0$ would be an inconsistent spurious constraint as it would also participate in the flexural strain field. Similarly, Thomas and Abbas (1975) presented a two noded beam element with w and θ expansions as in Eqs. (23) and (24), resulting in ISC.

The beam formulation of Heyliger and Reddy (1988) needs to be studied here, wherein the higher order terms are retained. The displacement model and the strain terms are given as

$$u=u_0+z\theta_x-kz^3(\theta_x+\beta_x) \tag{26}$$

$$w=w_0, \tag{27}$$

where

$$k=\frac{4}{3h^2},\ \beta_x=\partial w/\partial x, \tag{27a}$$

$$\varepsilon_x=u_{0,x}+z\theta_{x,x}-kz^3(\theta_{x,x}+\beta_{x,x}) \tag{28}$$

$$\gamma_{xz}=(1-3kz^2)(\theta_x+\beta_x). \tag{29}$$

If the displacements are expanded as

$$u=a_0+a_1x \tag{30}$$

$$\theta_x = b_0 + b_1 x \tag{31}$$

$$\beta_x = c_1 + 2c_2 x + 3c_3 x^2, \tag{32}$$

the strain fields can be rewritten as

$$\varepsilon_x = a_1 + zb_1 - kz^3(b_1 + 2c_2 + 6c_3 x) \tag{33}$$

$$\gamma_{xz} = (1 - 3kz^2)\left[(b_0 + c_1) + (b_1 + 2c_2)x + 3c_3 x^2\right], \tag{34}$$

where c_3 is an inconsistent constraint and would be spurious as it also participates in flexural strain, as in Eq. (33).

Similarly, a higher order C^1 continuous shear deformable plate element has been developed with full integration for evaluating shear strain energy without locking (Phan and Reddy, 1985). The inplane displacements (u,v) and face rotations (θ_x,θ_y) are interpolated linearly over a four-noded rectangular element, while transverse displacement is interpolated using Hermite cubic polynomial as shown below:

$$u_0 = a_0 + a_1 x + a_2 y + a_3 xy, \tag{35a}$$

$$v_0 = b_0 + b_1 x + b_2 y + b_3 xy, \tag{35b}$$

$$\theta_x = c_0 + c_1 x + c_2 y + c_3 xy, \tag{35c}$$

$$\theta_y = d_0 + d_1 x + d_2 y + d_3 xy \tag{35d}$$

$$w = e_0 + e_1 x + e_2 y + e_3 x^2 + e_4 xy + e_5 y^2 + e_6 x^3 + e_7 x^2 y + e_8 xy^2 + e_9 y^3 + e_{10} x^3 y + e_{11} xy^3 \tag{35e}$$

The displacement model is given as

$$u = u_0 + k_1\theta_x - kz^3\beta_x, \tag{36}$$

$$v = v_0 + k_1\theta_y - kz^3\beta_y \tag{37}$$

$$w = w_0, \tag{38}$$

where

$$k_1 = (z - kz^3),\ \beta_y = \partial w/\partial y \tag{38a}$$

and the strain components of the plate element in flexure can be expressed from the above as

$$\varepsilon_x = \left(a_1 + k_1 c_1 - 2kz^3 e_3\right) - 6kz^3 e_6 x + \left(a_3 + k_1 c_3 - 2kz^3 e_7\right)y - 6kz^3 e_{10} xy, \tag{39}$$

$$\varepsilon_y = \left(b_2 + k_1 d_2 - 2kz^3 e_5\right) - 6kz^3 e_9 y + \left(b_3 + k_1 d_3 - 2kz^3 e_8\right)x - 6kz^3 e_{11} xy, \tag{40}$$

$$\gamma_{xy} = \left(a_2 + b_1 + k_1(c_2 + d_1) - 2kz^3e_4\right) + \left(a_3 + k_1c_3 - 4kz^3e_7\right)x + \left(b_3 + k_1d_3 - 4kz^3e_8\right)y - 6kz^3e_{10}x^2 - 6kz^3e_{11}y^2, \tag{41}$$

$$\gamma_{xz} = k_2\left[(c_0 + e_1) + (c_1 + 2e_3)x + (c_2 + e_4)y + (c_3 + 2e_7)xy + 3e_6x^2 + e_8y^2 + 3e_{10}x^2y + e_{11}y^3\right] \tag{42}$$

$$\gamma_{yz} = k_2\left[(d_0 + e_2) + (d_1 + e_4)x + (d_2 + 2e_5)y + (d_3 + 2e_8)xy + e_7x^2 + 3e_9y^2 + 3e_{11}xy^2 + e_{10}x^3\right] \tag{43}$$

where

$$k_2 = (1 - 3kz^2). \tag{43a}$$

Now, the inconsistent constraints can be expressed as

$$e_i \rightarrow 0;\ i = 6,7\ldots 11. \tag{44}$$

Here, all these inconsistent constraints turn spurious due to their participation in other strain fields, as seen from Eqs. (39)–(41).

Balasubramanian and Prathap (1989) formulated an arch element wherein the inplane displacement u, radial displacement w and the face rotation θ were all expanded using cubic polynomials, retaining ISC.

In all these cases of anisoparametric elements, ISC do not directly lead to shear locking, but to delayed convergence and stress oscillations. Error models to predict these effects *a priori* (Prathap, 1993) and stress oscillations patterns (Balasubramanian and Prathap, 1989) can be seen in these works.

9.3. Consistent constraints (CC)

The consistency of constraints in anisoparametric elements are maintained either by choosing appropriate order of polynomials for the variables participating in the constrained strain field or by considering the constrained strain field itself as an independent degree of freedom.

The first approach has been adopted by Tessler and Dong (1981) and Nickel and Secor (1972). The latter modelled w using a cubic polynomial as in Eq. (23) and the face rotation θ with a quadratic polynomial as

$$\theta = b_0 + b_1x + b_2x^2, \tag{45}$$

resulting in shear strain as

$$\gamma = (a_1 + b_0) + (2a_2 + b_1)x + (3a_3 + b_2)x^2. \tag{46}$$

It can be seen that the constraints from this shear strain field would be consistent (Prathap, 1993) with coefficients from both w and θ participating in all of them.

The alternate approach of retaining the constrained shear angle itself as a nodal degree of freedom had been adopted by Thomas et al. (1973) and To (1981). The transverse displacement is represented by a cubic polynomial as in Eq. (23) and the shear angle γ using a linear one as

$$\gamma = b_0 + b_1x. \tag{47}$$

As the shear strain is retained as an independent entity and the shear energy is computed based only on γ (without any participation from other degrees of freedom), it remains consistent. The face rotation θ is then computed, in these cases, for evaluating bending energy as

$$\theta = w_{,x} + \gamma. \tag{48}$$

As a minor variation on this approach, Kapur (1966) treated the transverse displacement due to flexure and shear independently using cubic polynomials as (w_f as in Eq. (23)),

$$w_s = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \tag{49}$$

with the shear strain described as and shear energy evaluated from

$$\gamma = b_1 + 2b_2 x + 3b_3 x^2. \tag{50}$$

As this shear strain field is independent of the flexural field, it remains consistent.

10. Conclusions

An analysis of an anisoparametric shear deformable element has been carried out using ideas that have emerged from what are called the field-consistency, variational-correctness and best-fit strain prediction paradigms (Prathap, 1994). This has led to an important classification of field-inconsistent constraints into two classes such as '*spurious*' and '*nonspurious*'. The main advantage of such a classification is the availability of *a priori* knowledge about the behaviour of such anisoparametric formulations.

Moreover, these paradigms provide a powerful conceptual framework on which the analytical study of finite elements can be based. Error models derived from these concepts are then verified by actual digital computation to show that the performance of such element(s), which are somewhat unconventional in design, can be rationalized in a scientific manner.

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